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Almost splitting sets in integral domains ☆

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Abstract

Let A be an integral domain, S a saturated multiplicative subset of A, and $N(S) = \{0 \neq x \in A | (x, s)_v = A$ for all $s \in S\}$. Then S is called an *almost splitting set* if for each $0 \neq d \in A$, there is an integer $n = n(d) \geqslant 1$ such that $d^n = st$ for some $s \in S$ and $t \in N(S)$. Let B be an overring of A, X an indeterminate over B, $R = A + XB[X]$, and $D = A + X^2B[X]$. In this paper, we study almost splitting sets and show that D is an AGCD-domain if and only if R is an AGCD-domain and *char* $(A) \neq 0$. As a corollary, we have that D is an AGCD-domain if A is an integrally closed AGCD-domain, *char* $(A) \neq 0$, and $B = A_S$, where S is an almost splitting set of A.

MSC: 13A05; 13A15; 13B25

1. Introduction

Let D be an integral domain with quotient field K, S a saturated multiplicative subset of D, and $N(S) = \{0 \neq x \in D | (x, s)_v = D$ for all $s \in S\}$. Then S is called a *splitting set* if each $0 \neq d \in D$ may be written as $d = sa$ for some $s \in S$ and $a \in N(S)$. Following [5], we say that S is a *t-splitting set* if for each $0 \neq d \in D$, $dD = (AB)_t$ for some integral ideals A and B of D, where $A_t \cap sD = sA_t$ for all $s \in S$ and $B_t \cap S \neq \emptyset$. It is easy to see that a splitting set is a t-splitting set, but a t-splitting set need not be a splitting set (see Proposition 2.7). However, if $Cl(D) = 0$, then a t-splitting set S of D is a splitting set. For if $0 \neq d \in D$, then

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$dD_S \cap D$ is a t-invertible t-ideal of D [5, Corollary 2.3; 25, Lemma 3.17]; hence $dD_S \cap D$ is principal. Thus S is a splitting set [3, Theorem 2.2].

Now we have a very similar and interesting question. *"What are the properties of a t-splitting set S of D when $Cl(D)$ is torsion ?"* Fortunately, by an argument similar to the one given in the proof of the case when $Cl(D)=0$, we can show that for each $0 \neq d \in D$, there is an integer $n=n(d) \geqslant 1$ such that $d^n D_S \cap D$ is principal (see the proof of Corollary 2.4). (This is equivalent to the fact that $d^n = st$ for some $s \in S$ and $t \in N(S)$; see Lemma 2.2.) This type of multiplicative sets was introduced by Dumitrescu et al. [19] to study when $A + XB[X]$ is an integrally closed AGCD-domain, where B is an overring of A and X is an indeterminate over B. (Recall that an integral domain D is an *almost GCD-domain* (AGCD-domain) if for every $a, b \in D$, there is an integer $n=n(a,b) \geqslant 1$ such that $a^n D \cap b^n D$ is principal.) As in [6], we say that a saturated multiplicative subset S of D is an *almost splitting set* if for each $0 \neq d \in D$, there is an integer $n = n(d) \geqslant 1$ such that $d^n = st$ for some $s \in S$ and $t \in N(S)$. The purpose of this paper is to study almost splitting sets and determine when the subring $A + X^2B[X]$ of $A + XB[X]$ is an AGCD-domain.

Let S be a t-splitting set of an integral domain D, and let $\mathscr{T} = \{A_1 \cdots A_n \mid A_i = d_i D_S \cap D$ for some $0 \neq d_i \in D\}$. Then $D_S = \cap\{D_P \mid P \in t\text{-Max}(D)$ and $P \cap S = \emptyset\}$, $D_{\mathscr{T}} = \cap\{D_P \mid P \in t\text{-Max}(D)$ and $P \cap S \neq \emptyset\}$, and $D = D_S \cap D_{\mathscr{T}}$, where $D_{\mathscr{T}} = \{x \in K \mid xC \subseteq D$ for some $C \in \mathscr{T}\}$ [5, Lemma 4.2 and Theorem 4.3]. A t-splitting set S of D is a *t-complemented t-splitting set* if $D_{\mathscr{T}} = D_T$ for some multiplicative subset T of D, and the saturation of T is called the *t-complement of S*. It is known, and easily proved, that if S is a t-complemented t-splitting set, then $N(S)$ is the t-complement of S and $N(S)$ is also a t-complemented t-splitting set with t-complement $N(N(S)) = \overline{S}$, the saturation of S in D [5, p. 15]. A t-splitting set was introduced in [5] to show that $D^{(S)} = D + XD_S[X]$ is a PVMD if and only if D is a PVMD and S is a t-splitting set of D. (Recall that D is a *Prüfer v-multiplication domain* (PVMD) if every finite type v-ideal of D is t-invertible.)

In Section 2, we study almost splitting sets. In particular, we show that an almost splitting set is a t-complemented t-splitting set and that if S is an almost splitting set of D, then $Cl(D)$ is torsion if and only if $Cl(D_S)$ and $Cl(D_{N(S)})$ are both torsion. We also give an example of a t-complemented t-splitting set S of D which is not an almost splitting set such that both $Cl(D_S)$ and $Cl(D_{N(S)})$ are torsion, but $Cl(D)$ is not torsion. Let B be an overring of an integral domain A, X an indeterminate over B, $R = A + XB[X]$, and $D = A + X^2B[X]$. We prove in Section 3 that D is an AGCD-domain if and only if R is an AGCD-domain and $char\,(A) \neq 0$. As a corollary, we have that D is an AGCD-domain if A is an integrally closed AGCD-domain, $char\,(A) \neq 0$, and $B = A_S$, where S is an almost splitting set of A.

Throughout this paper, D is an integral domain with quotient field K, $U(D)$ is the group of units of D, and $char\,(D)$ is the characteristic of D. An *overring* of D means a ring between D and K. As usual, for $f \in K[X]$, the *content* A_f *of f* is the fractional ideal of D generated by the coefficients of f. Recall that for a nonzero fractional ideal I of D, $I^{-1} = \{x \in K \mid xI \subseteq D\}$, $I_v = (I^{-1})^{-1}$, and $I_t = \cup\{(a_1, \ldots, a_n)_v \mid (0) \neq (a_1, \ldots, a_n) \subseteq I\}$. We say that I is a *divisorial ideal* or *v-ideal* (resp., *t-ideal*) if $I = I_v$ (resp., $I = I_t$), while I_v is a *finite type v-ideal* if $I_v = (a_1, \ldots, a_n)_v$ for some $(0) \neq (a_1, \ldots, a_n) \subseteq I$. Let t-Max(D) be the set of ideals maximal among proper integral t-ideals of D. It is well known that (i) t-Max$(D) \neq \emptyset$ if D is not a field, (ii) every ideal in t-Max(D) is prime, (iii) $D = \cap_{P \in t\text{-Max}(D)} D_P$, and (iv) every prime ideal minimal over a t-ideal is a t-ideal, in particular, every height-one prime ideal is a t-ideal.

A nonzero fractional ideal I of D is said to be *t-invertible* if $(II^{-1})_t = D$. It is well known that the set $T(D)$ of t-invertible fractional t-ideals of D is an abelian group under the t-multiplication $I * J = (IJ)_t$. Let $Prin(D)$ be its subgroup of nonzero principal fractional ideals. We recall that as in [14,15], the (t-)*class group* of D is the quotient group $Cl(D) = T(D)/Prin(D)$. If D is a Krull domain, then $Cl(D)$ is just the divisor class group (see [21]). Many researchers have studied the class group of integral domains; for example, see [3,7–9,13,15,20,22].

Let S be a multiplicative subset of D. Then the set $N(S) = \{0 \neq t \in D | (s,t)_v = D$ for all $s \in S\}$ is a saturated multiplicative subset of D called the *m-complement of S*. It is clear that $S \cap N(S) \subseteq U(D)$ (equality holds if S is saturated) and that S is a splitting set if and only if S is saturated and $SN(S) = D \backslash \{0\}$. The reader is referred to [2,6,10] for the m-complement of a multiplicative set, to [2,3,5,10,16,18] for splitting or t-splitting sets, and to [8,11,12] for integral domains of the form $A + X^2B[X]$. Any undefined concepts or notation are standard as in [23,26].

2. Almost splitting sets

We begin this section with the following well-known results. The reader may consult [25] or Zafrullah's survey article [28] for the t-operation.

Lemma 2.1. *Let I be a nonzero fractional ideal of an integral domain D, and let S be a multiplicative subset of D.*

(1) *If I_t is of finite type, then $(ID_S)^{-1} = I^{-1}D_S$ and $(ID_S)_v = (I_vD_S)_v$. In particular, if I is t-invertible, then $(ID_S)_v = I_vD_S$.*
(2) *$(ID_S)_t = (I_tD_S)_t$ for any I.*
(3) *$(ID_S)_t \cap D$ is a t-ideal of D.*
(4) *I is t-invertible if and only if I_t is of finite type and I is t-locally principal.*
(5) *If I is a t-ideal, then $I = \cap_{P \in t\text{-Max}(D)} ID_P$.*
(6) *If $I = I_t \subsetneq D$ and $(I,s)_t = D$ for all $s \in S$, then $ID_S \cap D = I$.*

Proof. For (1) and (2), see [25, Lemma 3.4] or [28, Lemma 1.4]. Conditions (3)–(5) appear in [25, Corollary 2.7, Proposition 2.8(3), and Lemma 3.17]. (6) For $0 \neq x \in ID_S \cap D$, let $A = (I : x) = \{a \in D | ax \in I\}$. Then $I \subseteq A$ and $A \cap S \neq \emptyset$ because $x \in ID_S$, and so $A_t = D$. Note that $A_t = A$ since I is a t-ideal [23, Exercise 1, p. 406]. Hence $A = D$, and thus $x \in I$. The reverse inclusion is clear. $\square$

Let S be a saturated multiplicative subset of an integral domain D. Recall that S is a splitting set if and only if for each $0 \neq d \in D$, $dD_S \cap D$ is principal [3, Theorem 2.2] and that S is a t-splitting set if and only if for each $0 \neq d \in D$, $dD_S \cap D$ is t-invertible [5, Corollary 2.3]. The following lemma is the almost splitting set analog which appears in [6, Proposition 2.7]. We recall it for easy reference of the reader.

Lemma 2.2. *Let S be a saturated multiplicative subset of an integral domain D. Then S is an almost splitting set if and only if for each $0 \neq d \in D$, there is an integer $n = n(d) \geqslant 1$ such that $d^n D_S \cap D$ is principal.*

Our first result shows that an almost splitting set is a t-complemented t-splitting set. Hence "splitting set $\Rightarrow$ almost splitting set $\Rightarrow$ t-complemented t-splitting set $\Rightarrow$ t-splitting set". However, the converse implications do not hold; for example, see Proposition 2.7 and [5, p. 15].

Proposition 2.3. *An almost splitting set is a t-complemented t-splitting set.*

Proof. Let S be an almost splitting set of an integral domain D, and let $N(S) = \{0 \neq x \in D \mid (x, s)_v = D$ for all $s \in S\}$. We first show that S is a t-splitting set. By Anderson et al. [5, Corollary 2.3], we need only show that for each $0 \neq d \in D$, $dD_S \cap D$ is t-invertible. Let $A = dD_S \cap D$, and let $n = n(d) \geqslant 1$ be an integer such that $d^n = st$ for some $s \in S$ and $t \in N(S)$. Then $d^n D_S \cap D = stD_S \cap D = tD_S \cap D = tD$ by Lemma 2.1(6). If $d^n D_S \cap D = (A^n)_t$, then $(A^n)_t = tD$. So $(A^n)_t$, and hence A, is t-invertible. Thus it suffices to show that $d^n D_S \cap D = (A^n)_t$.

Since $A = dD_S \cap D$ and $d \in A$, we have $dD_S = AD_S$, and hence $d^n D_S = A^n D_S = (A^n D_S)_t \supseteq (A^n)_t D_S \supseteq A^n D_S$ by Lemma 2.1(2). So $d^n D_S = (A^n)_t D_S = A^n D_S$ and $(A^n)_t \subseteq d^n D_S \cap D$. For the reverse containment, let $x \in d^n D_S \cap D$ and $I = ((A^n)_t : x) = \{a \in D \mid ax \in (A^n)_t\}$. Then I is a t-ideal [23, Exercise 1, p. 406], and since $d^n D_S = (A^n)_t D_S$, we have $I \cap S \neq \emptyset$. Moreover, since $t \in d^n D_S \cap D \subseteq dD_S \cap D = A$, it follows that $t^n \in A^n \subseteq (A^n)_t \subseteq I$. Let $s \in I \cap S$. Then $D = (s, t^n)_v \subseteq I$, and thus $x \in xD = x(s, t^n)_v \subseteq xI \subseteq (A^n)_t$.

We next show that S is t-complemented. Let $\mathscr{T} = \{A_1 \cdots A_k \mid A_i = d_i D_S \cap D$ for some $0 \neq d_i \in D\}$; then $D_{\mathscr{T}} = \cap\{D_P \mid P \cap S \neq \emptyset$ and $P \in t\text{-Max}(D)\}$ [5, Lemma 4.2 and Theorem 4.3]. We claim that $D_{N(S)} = D_{\mathscr{T}}$. Clearly, $D_{N(S)} \subseteq D_{\mathscr{T}}$ since $tD_S \cap D = tD$ for all $t \in N(S)$ by Lemma 2.1(6). For the reverse containment, let $x \in D_{\mathscr{T}}$. Then $xA_1 \cdots A_k \subseteq D$ for some $A_1 \cdots A_k \in \mathscr{T}$. Since $A_i = d_i D_S \cap D$ for some $0 \neq d_i \in D$, there is an integer $m \geqslant 1$ such that $((A_1 \cdots A_k)^m)_t = aD$ for some $a \in N(S)$ (see the above paragraph). Hence $xa \in x((A_1 \cdots A_k)^m)_t \subseteq x(A_1 \cdots A_k)_t \subseteq D$, and thus $x \in D_{N(S)}$. $\square$

Corollary 2.4. *Let D be an integral domain with $Cl(D)$ torsion, and let S be a saturated multiplicative subset of D. Then S is an almost splitting set if and only if S is a t-splitting set.*

Proof. Assume that S is a t-splitting set, and let $0 \neq d \in D$. Then $dD = (AB)_t$ for some t-invertible integral ideals A and B of D such that $A_t \cap sD = sA_t$ for all $s \in S$ and $B_t \cap S \neq \emptyset$. Since $Cl(D)$ is torsion, there is an integer $n \geqslant 1$ such that $(A^n)_t = aD$ for some $0 \neq a \in D$. Clearly $(a, s)_v = D$ for all $s \in S$; so $aD_S \cap D = aD$ by Lemma 2.1(6). Since $dD = (AB)_t$, it follows that $d^n D_S = ((A^n B^n)_t D_S)_t = ((A^n)_t D_S)_t$ (Lemma 2.1(2)). So $d^n D_S \supseteq (A^n)_t D_S \supseteq (A^n B^n)_t D_S = d^n D_S$, or $d^n D_S = (A^n)_t D_S$. Hence $d^n D_S \cap D = (A^n)_t D_S \cap D = aD_S \cap D = aD$. Thus S is an almost splitting set by Lemma 2.2. The converse always holds by Proposition 2.3. $\square$

Remark 2.5. Let D be an integral domain, X an indeterminate over D, and $\emptyset \neq S \subseteq \{f \in D[X] | (A_f)_v = D\}$ a saturated multiplicative subset of $D[X]$. In [16, Proposition 3.7], we showed that S is a t-complemented t-splitting set. Note that $Cl(D) = Cl(D[X])$ if and only if D is integrally closed [22, Theorem 3.6]. Thus if D is an integrally closed domain with $Cl(D)$ torsion, then S is an almost splitting set by Corollary 2.4.

Recall that an integral domain D is a GCD-domain (resp., UMT-domain) if and only if $D\backslash\{0\}$ is a splitting set (resp., t-splitting set) in $D[X]$ [3, Example 4.7] (resp. [16, Corollary 2.9]). (An integral domain D is called a *UMT-domain* if every upper to zero in $D[X]$ is a maximal t-ideal. It is well known that if D is an integrally closed UMT-domain if and only if D is a PVMD [24, Proposition 3.2].) We next give the almost splitting set analog.

Proposition 2.6. *Let D be an integrally closed domain and X an indeterminate over D. Then $D\backslash\{0\}$ is an almost splitting set in $D[X]$ if and only if D is an AGCD-domain.*

Proof. Recall that an integrally closed domain D is an AGCD-domain if and only if D is a PVMD with $Cl(D)$ torsion [27, Corollary 3.8 and Theorem 3.9].

($\Rightarrow$) Suppose that $D\backslash\{0\}$ is an almost splitting set in $D[X]$, and let $0 \neq f \in D[X]$. Then there is an integer $n = n(f) \geqslant 1$ such that $f^n = ag$ for some $0 \neq a \in D$ and $g \in D[X]$ with $(d, g)_v = D[X]$ for all $0 \neq d \in D$. Clearly, $(A_g)_v = D$; hence $(A_f^n)_v = (A_{f^n})_v = (A_{ag})_v = aD$ as D is integrally closed [23, Proposition 34.8]. Thus A_f is t-invertible, which implies that D is a PVMD. Moreover, since $(A_f^n)_v$ is principal, we can conclude that $Cl(D)$ is torsion.

($\Leftarrow$) Assume that D is an AGCD-domain, and let $0 \neq f \in D[X]$. Then there is an integer $n = n(f) \geqslant 1$ such that $(A_f^n)_v = aD$ for some $a \in D$; so $(A_{f^n})_v = aD$ [23, Proposition 34.8] because D is integrally closed. Let $g = f^n/a$. Then $f^n = ag$ and $g \in D[X]$ with $(A_g)_v = D$; so $(d, g)_v = D[X]$ for all $0 \neq d \in D$ [24, Proposition 1.1]. Thus $D\backslash\{0\}$ is an almost splitting set. $\square$

We next give an example of a t-complemented t-splitting set which is not an almost splitting set.

Proposition 2.7. *Let D be an integral domain, X an indeterminate over D, and $S = \{uX^n | u \in U(D)$ and $n = 0, 2, 3, \ldots\}$. Then*:

(1) *S is a saturated multiplicative subset of $D[X^2, X^3]$.*
(2) *S is a t-complemented t-splitting set of $D[X^2, X^3]$ and the t-complement of S is $D[X^2, X^3]\backslash X^2D[X]$.*
(3) *S is an almost splitting set of $D[X^2, X^3]$ if and only if $char\,(D) \neq 0$.*
(4) *S is not a splitting set of $D[X^2, X^3]$.*

Proof. Recall that $X^2D[X]$ is a height-one maximal t-ideal of $D[X^2, X^3]$ and if Q is a maximal t-ideal of $D[X^2, X^3]$, then either $Q = X^2D[X]$ or $Q \cap S = \emptyset$ [8, Lemma 1]. Also, note that $D[X^2, X^3]_S = D[X, X^{-1}] = D[X]_S$.

(1) This is clear.

(2) We first show that S is a t-splitting set. To do this, it suffices to show that for each $0 \neq f \in D[X^2, X^3]$, $fD[X]_S \cap D[X^2, X^3]$ is t-invertible [5, Corollary 2.3]. Let $I = fD[X]_S \cap D[X^2, X^3]$. Then $ID[X]_S = fD[X]_S$, $I \nsubseteq X^2D[X]$ (note that $(a + Xg)(a - Xg) = a^2 - X^2g^2 \in D[X^2, X^3]$ for all $a \in D$ and $g \in D[X]$), and I is a t-ideal of $D[X^2, X^3]$ by Lemma 2.1(3).

Let Q be a maximal t-ideal of $D[X^2, X^3]$. If $Q = X^2D[X]$, then $ID[X^2, X^3]_Q = D[X^2, X^3]_Q$. Assume that $Q \neq X^2D[X]$. Then $Q \cap S = \emptyset$, and so $ID[X^2, X^3]_Q = (ID[X^2, X^3]_S)_{Q_S} = (fD[X^2, X^3]_S)_{Q_S} = fD[X^2, X^3]_Q$. Thus I is t-locally principal. Hence if I is of finite type, then I is t-invertible by Lemma 2.1(4). Let $g \in I \backslash X^2D[X]$. Then $fD[X^2, X^3]_S \subseteq (g, X^2f)_v D[X^2, X^3]_S \subseteq ID[X^2, X^3]_S = fD[X^2, X^3]_S$; so $(g, X^2f)_v D[X^2, X^3]_S = ID[X^2, X^3]_S$. Hence $I_Q = ((g, X^2f)_v)_Q$ for all maximal t-ideals Q of $D[X^2, X^3]$, and thus $I = (g, X^2f)_v$ by Lemma 2.1(5).

We next show that S is t-complemented. Let Q be a maximal t-ideal of $D[X^2, X^3]$ such that $Q \cap S \neq \emptyset$. Then $Q = X^2D[X]$, and hence $\cap\{D_Q | Q \cap S \neq \emptyset$ and $Q \in t\text{-Max}(D)\} = D[X^2, X^3]_{X^2D[X]}$. Thus S is t-complemented with t-complement $D[X^2, X^3]\backslash X^2D[X]$.

(3) ($\Rightarrow$) Assume that S is an almost splitting set, and let $f = X^2(1 + X)$. Then $f \in D[X^2, X^3]$, and since S is an almost splitting set, there is an integer $n = n(f) \geqslant 1$ such that $f^nD[X]_S \cap D[X^2, X^3] = gD[X^2, X^3]$ for some $0 \neq g \in D[X^2, X^3]$ by Lemma 2.2. It is clear that $g(0) \neq 0$, $f^nD[X]_S = gD[X]_S$, and $f^n \in gD[X^2, X^3]$. So $f^n = uX^mg$ for some $u \in U(D)$ and integer $m \geqslant 0$, and hence $(1 + X)^n = ug$ because $g(0) \neq 0$. Note that $g \in D[X^2, X^3]$ and $(1 + X)^n = 1 + nX + [n(n + 1)/2]X^2 + \cdots + X^n$; so $nX = 0$. Thus $char\ (D) \neq 0$.

($\Leftarrow$) Assume that char$(D) = p \neq 0$, and let $0 \neq f = X^ng \in D[X^2, X^3]$, where $n \geqslant 0$ is an integer and $g \in D[X]$ with $g(0) \neq 0$. Then $g^p \in D[X^2, X^3]$ and $f^pD[X]_S = g^pD[X]_S$. If $h \in D[X]$ such that $g^ph \in D[X^2, X^3]$, then $h \in D[X^2, X^3]$ because $g^p(0) \neq 0$ and $g^p \in D[X^2, X^3]$. So $f^pD[X]_S \cap D[X^2, X^3] = g^pD[X]_S \cap D[X^2, X^3] = g^pD[X^2, X^3]$. Thus by Lemma 2.2, S is an almost splitting set.

(4) Let $f = X^2(1 + X) \in D[X^2, X^3]$. Then $fD[X]_S \cap D[X^2, X^3]$ is not principal, and thus S is not a splitting set [3, Theorem 2.2]. □

Corollary 2.8 (*cf. Anderson et al. [11, Theorem 2.5]*). *Let D be an integral domain, X an indeterminate over D, and $S = \{uX^n | u \in U(D)$ and $n = 0, 2, 3, \ldots\}$. Let I be a nonzero integral ideal of $D[X]$ such that $ID[X]_S \cap D[X] = I$. Then I is a t-ideal of $D[X]$ if and only if $I \cap D[X^2, X^3]$ is a t-ideal of $D[X^2, X^3]$.*

Proof. Let $T = \{uX^n | u \in U(D)$ and $n = 0, 1, 2, \ldots\}$, and note that $D[X]_S = D[X]_T = D[X^2, X^3]_S$.

($\Rightarrow$) Assume that I is a t-ideal of $D[X]$. Then $ID[X]_T = ID[X]_S$ is a t-ideal of $D[X]_S$ [3, Corollary 3.5] since T is a splitting set in $D[X]$ [3, Example 4.5], and hence $I \cap D[X^2, X^3] = (ID[X]_S \cap D[X]) \cap D[X^2, X^3] = ID[X]_S \cap D[X^2, X^3]$ is a t-ideal of $D[X^2, X^3]$ by Lemma 2.1(3). ($\Leftarrow$) Assume that $I \cap D[X^2, X^3]$ is a t-ideal of $D[X^2, X^3]$, and let $J = I \cap D[X^2, X^3]$. Then $JD[X]_S = ID[X]_S$ and $JD[X]_S$ is a t-ideal of $D[X]_S$ [5, Theorem 4.9] since S is a t-splitting set in $D[X^2, X^3]$ (Proposition 2.7(2)). Thus $I = ID[X]_S \cap D[X] = JD[X]_S \cap D[X]$ is a t-ideal of $D[X]$ (Lemma 2.1(3)). □

It is well known that if S is a splitting set of an integral domain D, then $Cl(D)=Cl(D_S)\oplus Cl(D_{N(S)})$ [3, Corollary 3.8]. This result cannot be generalized to a t-complemented t-splitting set [5, Remark 4.13]. We next give an example which shows that [3, Corollary 3.8] cannot be extended to an almost splitting set.

Example 2.9. Let D be an integral domain with quotient field K, X an indeterminate over D, $S=\{uX^n|u\in U(D)$ and $n=0,2,3,\ldots\}$, and $N(S)=\{f\in D[X^2,X^3]|(f,uX^n)_v=D[X^2,X^3]$ for all $uX^n\in S\}$. Then S is a t-complemented t-splitting set(in particular, an almost splitting set if char$(D)\neq 0$) in $D[X^2,X^3]$ and $N(S)=D[X^2,X^3]\backslash X^2D[X]$ by Proposition 2.7(2). Recall from [8, Lemma 1(1)] that $D[X^2,X^3]_{N(S)}$ is one-dimensional quasilocal, and thus $Cl(D[X^2,X^3]_{N(S)})=0$ and $Cl(D[X^2,X^3]_S)\oplus Cl(D[X^2,X^3]_{N(S)})$ $=Cl(D[X,X^{-1}])$ $=Cl(D[X])$ [3, Example 4.5]. Therefore, $Cl(D[X^2,X^3])\neq Cl(D[X^2,X^3]_S)\oplus Cl(D[X^2,X^3]_{N(S)})$ because $Cl(D[X^2,X^3])=Cl(D[X])\oplus K$, where K is considered as an additive abelian group[8, Theorem 6].

While we cannot generalize the nice property of splitting sets for the class group to almost splitting sets, we have the following useful result for the class group of almost splitting sets.

Theorem 2.10. *Let D be an integral domain, S an almost splitting set of D, and $N(S)=\{0\neq t\in D|(s,t)_v=D$ for all $s\in S\}$.*

(1) *If I is a t-invertible integral t-ideal of D, then there is an integer $n\geqslant 1$ such that $(I^n)_t=((S_1)(N_1))_t=(S_1)_t\cap(N_1)_t$ for some $\emptyset\neq S_1\subseteq S$ and $\emptyset\neq N_1\subseteq N(S)$.*
(2) *$Cl(D)$ is torsion if and only if $Cl(D_S)$ and $Cl(D_{N(S)})$ are torsion.*

Proof. (1) Let $I=(a_1,\ldots,a_k)_v$. Then there is an integer $n\geqslant 1$ such that $a_i^n=s_it_i$ for some $s_i\in S$ and $t_i\in N(S)$. Since I is t-invertible, $(I^n)_t=(a_1^n,\ldots,a_k^n)_v=(s_1t_1,\ldots,s_kt_k)_v$ [1, Lemma 3.3]. Let Q be a maximal t-ideal of D. Then since $Q\cap S=\emptyset$ or $Q\cap N(S)=\emptyset$, we have $(s_1t_1,\ldots,s_kt_k)_Q=((s_1,\ldots,s_k)(t_1,\ldots,t_k))_Q$. So by Lemma 2.1(1),

$$\begin{aligned}((I^n)_t)_Q&=((s_1t_1,\ldots,s_kt_k)_v)_Q=((s_1t_1,\ldots,s_kt_k)_Q)_v\\&=(((s_1,\ldots,s_k)(t_1,\ldots,t_k))_Q)_v\supseteq(((s_1,\ldots,s_k)(t_1,\ldots,t_k))_v)_Q\\&\supseteq((s_1t_1,\ldots,s_kt_k)_v)_Q=((I^n)_t)_Q.\end{aligned}$$

Hence $((s_1t_1,\ldots,s_kt_k)_v)_Q=(((s_1,\ldots,s_k)(t_1,\ldots,t_k))_v)_Q$ for all maximal t-ideals Q of D. Thus $(I^n)_t=((s_1,\ldots,s_k)(t_1,\ldots,t_k))_v=(s_1,\ldots,s_k)_v\cap(t_1,\ldots,t_k)_v$ by Lemma 2.1(5) and the fact that $((s_1,\ldots,s_k)+(t_1,\ldots,t_k))_v=D$.

(2) ($\Rightarrow$) Recall that almost splitting sets are t-complemented t-splitting sets (Proposition 2.3); hence the map $\varphi: Cl(D)\to Cl(D_S)\oplus Cl(D_{N(S)})$, given by $[I]\to([ID_S],[ID_{N(S)}])$, is surjective [5, Remark 4.13]. Thus if $Cl(D)$ is torsion, then $Cl(D_S)\oplus Cl(D_{N(S)})$, and hence both $Cl(D_S)$ and $Cl(D_{N(S)})$, are torsion.

($\Leftarrow$) Assume that $Cl(D_S)$ and $Cl(D_{N(S)})$ are both torsion, and let I be a t-invertible integral t-ideal of D. Then ID_S and $ID_{N(S)}$ are t-invertible, and thus there exists an integer $n\geqslant 1$ such that $((ID_S)^n)_t=(I^n)_tD_S=aD_S$ and $((ID_{N(S)})^n)_t=(I^n)_tD_{N(S)}=bD_S$ for some $a,b\in D$ (see Lemma 2.1(1) for the equalities). Since $(I^n)_t$ is a t-invertible t-ideal and S is an almost splitting set, by (1) we can choose another integer $m \geqslant 1$ such that $(I^{nm})_t = (((I^n)_t)^m)_t = ((S_1)(N_1))_t$, $a^m = s't$, and $b^m = st'$ for some $\emptyset \neq S_1 \subseteq S$, $\emptyset \neq N_1 \subseteq N(S)$, $s, s' \in S$, and $t, t' \in N(S)$. Also, since I is t-invertible, by Lemma 2.1(1) $(N_1)_t D_S = (I^{nm})_t D_S = ((I^n)_t D_S)^m)_t = a^m D_S = t D_S$ and $(S_1)_t D_{N(S)} = (I^{nm})_t D_{N(S)} = ((I^n)_t D_{N(S)})^m)_t = b^m D_{N(S)} = s D_{N(S)}$. Therefore, $(I^{nm})_t = ((S_1)(N_1))_t = (S_1)_t \cap (N_1)_t = ((S_1)_t D_{N(S)} \cap D) \cap ((N_1)_t D_S \cap D) = (s D_{N(S)} \cap D) \cap (t D_S \cap D) = sD \cap tD = st D$ by Lemma 2.1(6). This means that $Cl(D)$ is torsion.

Let S be a t-complemented t-splitting set of an integral domain D. As we noted in the proof of ($\Rightarrow$) of Theorem 2.10(2), if $Cl(D)$ is torsion, then $Cl(D_S)$ and $Cl(D_{N(S)})$ are both torsion (or see [5, Remark 4.13]). Our next example shows that the converse does not hold.

Example 2.11. Let the notation be as in Example 2.9. Assume that D is an integrally closed domain with $Cl(D)$ torsion. Then $Cl(D[X^2, X^3]_{N(S)}) = 0$ and $Cl(D[X^2, X^3]_S) = Cl(D[X]) = Cl(D)$ is torsion [22, Theorem 3.6]. But since $Cl(D[X^2, X^3]) = Cl(D) \oplus K$ [8, Corollary 7], $Cl(D[X^2, X^3])$ is not torsion if and only if $char\ (K) = 0$, if and only if S is not an almost splitting set (cf. Proposition 2.7(3) and Theorem 2.10(2)). For example, if $D = \mathbb{Z}$ is the ring of integers, then $Cl(\mathbb{Z}[X^2, X^3]_{N(S)}) = Cl(\mathbb{Z}[X^2, X^3]_S) = 0$ but $Cl(\mathbb{Z}[X^2, X^3]) = \mathbb{Q}$ is torsion-free, where $\mathbb{Q}$ is the additive group of rational numbers.

Let D be an integral domain and $X^1(D)$ the set of height-one prime ideals of D. Then D is called a *weakly Krull domain* if $D = \cap_{P \in X^1(D)} D_P$ and the intersection has finite character. Recall that D is an *almost weakly factorial domain* (AWFD) if for each nonzero nonunit $x \in D$, some positive power of x is a product primary elements. It is known that D is an AWFD if and only if D is a weakly Krull domain and $Cl(D)$ is torsion [4, Theorem 3.4]. For more on weakly Krull domains and AWFD's, see [4,12].

Corollary 2.12. *Let S be an almost splitting set of an integral domain D and $N(S) = \{0 \neq x \in D | (x, s)_v = D$ for all $s \in S\}$.*

(1) *D is an AGCD-domain if and only if D_S and $D_{N(S)}$ are AGCD-domains.*
(2) *D is weakly Krull if and only if D_S and $D_{N(S)}$ are weakly Krull.*
(3) *D is an AWFD if and only if D_S and $D_{N(S)}$ are AWFDs.*

Proof. (1) Assume that both D_S and $D_{N(S)}$ are AGCD-domains, and let $0 \neq a, b \in D$. Then as S is an almost splitting set, there is an integer $n \geqslant 1$ such that $a^n = s_1 t_1$ and $b^n = s_2 t_2$ for some $s_i \in S$ and $t_i \in N(S)$. By assumption and [27, Lemma 3.6], there is another integer $m \geqslant 1$ such that $s_1^m D_{N(S)} \cap s_2^m D_{N(S)} = s D_{N(S)}$ and $t_1^m D_S \cap t_2^m D_S = t D_S$ for some $s, t \in D$. Recall that for any $0 \neq x, y, d \in D$, if $x D_{N(S)} \cap y D_{N(S)} = d D_{N(S)}$, then $x^k D_{N(S)} \cap y^k D_{N(S)} = d^k D_{N(S)}$ for all integers $k \geqslant 1$ [27, Lemma 3.6]. Thus as S and $N(S)$ are almost splitting sets, we may assume that $s \in S$ and $t \in N(S)$. So $s_1^m D \cap s_2^m D = (s_1^m D_{N(S)} \cap D) \cap (s_2^m D_{N(S)} \cap D) = s D_{N(S)} \cap D = sD$ by Lemma 2.1(6). Similarly, $t_1^m D \cap t_2^m D = tD$. Thus $a^{nm} D \cap b^{nm} D = s_1^m t_1^m D \cap s_2^m t_2^m D = s_1^m D \cap s_2^m D \cap t_1^m D \cap t_2^m D = sD \cap tD = stD$. The converse always holds for any multiplicative subset of D.

(2) It is well known that if D is weakly Krull, then D_N is also weakly Krull for any multiplicative subset N of D. The converse follows directly from the fact that $D=D_S\cap D_{N(S)}$ [2, Proposition 1.1].

(3) This is an immediate consequence of (2) and Theorem 2.10(2) since D is an AWFD if and only if D is weakly Krull and $Cl(D)$ is torsion [4, Theorem 3.4]. □

3. AGCD-domains of the form $A + X^2B[X]$

Let B be an overring of an integral domain A, X an indeterminate over A, and $R = A + XB[X]$. In [19, Theorem 3.1], the authors showed that R is an integrally closed AGCD-domain if and only if A is an integrally closed AGCD-domain and $B = A_S$, where S is an almost splitting set in A. They also gave some examples of non-integrally closed AGCD-domains. For example, if A is an integrally closed AGCD-domain of $char\ (A)=p \neq 0$ such that $A \neq A^p$, then $A[X^p, X^{p+1}, \ldots, X^{2p+1}]$ and $A^p + XA[X]$ are non-integrally closed AGCD-domains. The purpose of this section is to prove that the domain $A + X^2B[X]$ is an AGCD-domain if and only if $A + XB[X]$ is an AGCD-domain and char$(A) \neq 0$. Using this result and [19, Theorem 3.1], we can construct simple examples of non-integrally closed AGCD-domains.

Let $A \subseteq B$ be an extension of integral domains. Following [17], we say that B is *t-linked over A* if $I^{-1} = A$ for a nonzero finitely generated ideal I of A implies $(IB)^{-1} = B$; equivalently, if P is a maximal t-ideal of B, then $(P \cap A)_t \subsetneq A$. Recall that A is of *finite t-character* if each nonzero nonunit element of A belongs to only finitely many maximal t-ideals of A. Examples of integral domains of finite t-character include Krull domains, Mori domains, Noetherian domains, and one-dimensional semi-quasilocal domains.

Let $A \subseteq B$ be an extension of integral domains, X an indeterminate over A, $R = A + XB[X]$, and $D = A + X^2B[X]$. In [12, Lemma 4.1], the authors proved that the map $Spec(R) \to Spec(D)$, given by $Q \mapsto Q\cap D$, is an order-preserving bijection. In particular, if $A = B$, then the bijection preserves t-ideals, i.e., Q is a prime t-ideal of R if and only if $Q \cap D$ is a prime t-ideal of D [11, Theorem 2.5] (or see Corollary 2.8). We next show that this holds for maximal t-ideals when B is an overring of A.

Lemma 3.1. *Let B be an overring of an integral domain A, X an indeterminate over A, $R=A+XB[X]$, $D=A+X^2B[X]$, and Q a nonzero prime ideal of R. Then Q is a maximal t-ideal of R if and only if $Q \cap D$ is a maximal t-ideal of D. In particular, R is t-linked over D, and R is of finite t-character if and only if D is of finite t-character*.

Proof. Recall that the map $Spec(R) \to Spec(D)$, given by $Q \mapsto Q \cap D$, is an order-preserving bijection [12, Lemma 4.1]. So we need only show that if Q is a maximal t-ideal of R, then $Q \cap D$ is a t-ideal of D and that if $Q \cap D$ is a maximal t-ideal of D, then Q is a t-ideal of R. (This means that Q is a maximal t-ideal of R if and only if $Q \cap D$ is a maximal t-ideal of D.)

Let K be the quotient field of A, Q a nonzero prime ideal of R, $P=Q\cap D$, and $S=\{X^n | n=0, 2, 3, \ldots\}$. Note that $Q\cap A=P\cap A$; $Q\cap S=\emptyset \Leftrightarrow P\cap S=\emptyset$; $R_S=B[X, X^{-1}]=B[X]_S=D_S$; and $PB[X]_S = QB[X]_S$.

Case 1: $Q \cap A = (0)$ ($\Leftrightarrow$ $P \cap A = (0)$). Note that $R_{A\setminus\{0\}} = K[X]$, $D_{A\setminus\{0\}} = K[X^2, X^3]$, and $dim(K[X]) = dim(K[X^2, X^3]) = 1$; so $ht\ Q = ht\ P = 1$. Thus Q and P are prime t-ideals of R and D, respectively.

Case 2: $Q \cap A \neq (0)$ and $Q \cap S = \emptyset$ ($\Leftrightarrow$ $P \cap A \neq (0)$ and $P \cap S = \emptyset$).

Assume that $(PB[X]_S)_t = (QB[X]_S)_t = B[X]_S$. Then there is a finitely generated subideal I of Q such that $(IB[X]_S)_v = B[X]_S$. Note that for any $0 \neq a \in Q \cap A$, $(I, a) \subseteq Q$, $((I, a)B[X]_S)_v = B[X]_S$, and (I, a) is finitely generated. Replacing I with (I, a), we may assume that $I \cap A \neq (0)$. So $(R : I) \subseteq K[X]$. Since $R \subseteq B[X]_S$, it follows that $(R : I) \subseteq (B[X]_S : I) = B[X]_S$, and thus $(R : I) \subseteq B[X]_S \cap K[X] = B[X]$. Hence $XB[X] \subseteq (R : B[X]) \subseteq (R : (R : I)) = I_v$, and thus $X \in I_v \subseteq Q_t$. Therefore, if Q is a t-ideal, then $(PB[X]_S)_t = (QB[X]_S)_t \subsetneq B[X]_S$. Similarly, if P is a prime t-ideal, then $(QB[X]_S)_t = (PB[X]_S)_t \subsetneq B[X]_S$.

Assume that Q or P is a maximal t-ideal. Then $(PB[X]_S)_t = (QB[X]_S)_t \subsetneq B[X]_S$ by the above paragraph, and hence $PB[X]_S = (PB[X]_S)_t = (QB[X]_S)_t = QB[X]_S$ (cf. Lemma 2.1(3)). Thus $Q = QB[X]_S \cap R$ and $P = PB[X]_S \cap D$ are t-ideals by Lemma 2.1(3).

Case 3: $Q \cap A \neq (0)$ and $Q \cap S \neq \emptyset$ ($\Leftrightarrow$ $P \cap A \neq (0)$ and $P \cap S \neq \emptyset$). It is clear that $Q = (Q \cap A) + XB[X]$ and $P = (Q \cap A) + X^2B[X]$. We first show that ($\#_1$) if I is a nonzero (integral) ideal of R such that $I \cap A \neq (0)$ and $X^2 \in I$, then $I_v = (I_0^{-1} \cap B)^{-1} \cap B + XB[X]$, where $I_0 = \{f(0) | f \in I\}$. Let $\omega \in I^{-1} = (R : I)$. Then $\omega \in B[X]$ because $I \cap A \neq (0)$ and $X^2 \in I$. Note that since $f\omega \in A + XB[X]$ for any $f \in I$, we have $f(0)\omega(0) \in A$, and so $\omega(0) \in I_0^{-1} \cap B$. Hence $I^{-1} = (I_0^{-1} \cap B) + XB[X]$ since $XB[X] \subseteq I^{-1}$. The same argument also shows that $I_v = (I^{-1})^{-1} = (I_0^{-1} \cap B)^{-1} \cap B + XB[X]$. Similarly, we can show that ($\#_2$) if J is a nonzero (integral) ideal of D such that $J \cap A \neq (0)$ and $X^2 \in J$, then $J_v = (J_0^{-1} \cap B)^{-1} \cap B + X^2B[X]$,where $J_0 = \{g(0) | g \in J\}$.

Assume that Q is a t-ideal of R, and let J be a finitely generated subideal of P. Note that $X^2 \in P$ and $J \subseteq (J, a, X^2) \subseteq P$ for any $0 \neq a \in A \cap Q$. So replacing J with (J, a, X^2), we may assume that $J \cap A \neq (0)$ and $X^2 \in J$. By ($\#_1$) and ($\#_2$), $(JR)_v = (J_0^{-1} \cap B)^{-1} \cap B + XB[X]$ and $J_v = (J_0^{-1} \cap B)^{-1} \cap B + X^2B[X]$ (note that $J_0 = \{g(0) | g \in J\} = \{f(0) | f \in JR\}$). Since Q is a t-ideal and JR is a finitely generated subideal of Q, $(J_0^{-1} \cap B)^{-1} \cap B \subseteq Q \cap A$, and thus $J_v \subseteq (Q \cap A) + X^2B[X] = P$, which implies that P is a t-ideal.

We next assume that P is a t-ideal, and let I be a finitely generated subideal of Q. As in the above paragraph, we may assume that $I \cap A \neq (0)$ and $X^2 \in I$. Let $I_0 = \{f(0) | f \in I\}$. Then $I_0 \neq (0)$ and I_0 is a finitely generated subideal of $Q \cap A$ because I is finitely generated and $XB[X] \subseteq Q$. Note that $(I_0, X^2)D$ is a finitely generated subideal of P such that $\{g(0) | g \in (I_0, X^2)D\} = I_0$, $(I_0, X^2)D \cap A \neq (0)$, and $X^2 \in (I_0, X^2)D$. So by ($\#_1$) and ($\#_2$), $((I_0, X^2)D)_v = (I_0^{-1} \cap B)^{-1} \cap B + X^2B[X]$ and $I_v = (I_0^{-1} \cap B)^{-1} \cap B + XB[X]$. Since P is a t-ideal, $(I_0^{-1} \cap B)^{-1} \cap B \subseteq Q \cap A$, and thus $I_v \subseteq (Q \cap A) + XB[X] = Q$. This shows that Q is a t-ideal.

Let $A \subseteq B$ be an extension of integral domains. Then B is said to be a *root extension* of A if for each $x \in B$, $x^n \in A$ for some integer $n \geqslant 1$.

Lemma 3.2. *Let $A \subseteq B$ be an extension of integral domains, X an indeterminate over B, $R = A + XB[X]$, and $D = A + X^2B[X]$. Then R is a root extension of D if and only if $char\ (A) \neq 0$.*

Proof. Assume that R is a root extension of D. Then $(1 + X)^n \in D$ for some integer $n \geqslant 1$. Hence $nX = 0$, and thus $char\ (A) \neq 0$. Conversely, if $char\ (A) = p \neq 0$, then for any $f \in R$, $f^p \in D$. Thus R is a root extension of D.

We next give the main result of this section. This result combined with [19, Theorem 3.1(a)] gives many examples of non-integrally closed AGCD-domains (see Corollary 3.5).

Theorem 3.3. *Let B be an overring of an integral domain A, X an indeterminate over A, $R = A + XB[X]$, and $D = A + X^2B[X]$. Then R is an AGCD-domain and $char\ (A) \neq 0$ if and only if D is an AGCD-domain.*

Proof. ($\Rightarrow$) Assume that R is an AGCD-domain and $char\ (A) = p \neq 0$. We first note that (#) if $f \in D$ with $f(0) \neq 0$, then $fR \cap D = fD$. For if $g = a_0 + a_1X + \cdots + a_nX^n \in R$ such that $fg \in D$, then $f(0)a_1 = 0$ since $fg = f(0)a_0 + f(0)a_1X + X^2g_1$ for some $g_1 \in B[X]$; so $a_1 = 0$. Hence $g \in D$, and thus $fR \cap D = fD$.

Let $0 \neq f, g \in D$.

Case 1: $f(0) \neq 0$ and $g(0) \neq 0$. Since R is an AGCD-domain, there is an integer $n = n(f, g) \geqslant 1$ such that $f^nR \cap g^nR = hR$ for some $h \in R$. Note that $f^n(0) \neq 0$, $g^n(0) \neq 0$, and $char\ A = p$; hence $h(0) \neq 0$ and $h^p \in D$. Thus $f^{np}D \cap g^{np}D = (f^{np}R \cap D) \cap (g^{np}R \cap D) = (f^{np}R \cap g^{np}R) \cap D = h^pR \cap D = h^pD$ by (#) and [27, Lemma 3.6].

Case 2: $f(0) \neq 0$ and $g = X^mg_1$, where $m \geqslant 2$ and $g_1 \in B[X]$ with $g_1(0) \neq 0$. Let $0 \neq s \in A$ such that $sg_1 \in R$ (note that B is an overring of A). Replacing f, g, g_1, and m with $(sf)^p$, $(sg)^p = X^{mp}(sg_1)^p$, $(sg_1)^p$, and mp, respectively, we may assume that $g_1 \in D$. Thus by the proof of Case 1, $f^nR \cap g_1^nR = hR$ and $f^nD \cap g_1^nD = hD$ for some integer $n \geqslant 1$ and $h \in D$ with $h(0) \neq 0$.

Note that R is an AGCD-domain and that for any integer $k \geqslant 1$, if $f^kR \cap g^kR$ is principal, then $f^{nk}R \cap g^{nk}R$ is also principal [27, Lemma 3.6]. So we may assume that $f^nR \cap g^nR = bR$ for some $b \in R$. Since $X^{nm}h \in f^nR \cap g^nR$, we have $X^{nm}h = bc$ for some $c \in R$. Also, since $b \in f^nR \cap g_1^nR = hR$, we have $b = hd$ for some $d \in R$. Hence $X^{nm}h = hdc$, and so $X^{nm} = dc$. Finally, since $b \in g^nR$, we have $b = g^nr = X^{nm}g_1^nr$ for some $r \in R$, and so $h = g_1^nrc$. Hence $c \in U(A)$ as $h(0) \neq 0$, and thus $f^nR \cap g^nR = X^{nm}hR$.

Let $g^nh_1 \in f^nD \cap g^nD$, where $h_1 \in D$. Then $g^nh_1 = X^{nm}h\alpha$ for some $\alpha \in R$ by the above paragraph; so $(g_1)^nh_1 = h\alpha$. Thus by (#), $\alpha \in D$ because $g_1, h_1 \in D$ and $h(0) \neq 0$, and hence $f^nD \cap g^nD \subseteq X^{nm}hD$. The reverse containment follows directly from the fact that $f^nD \cap g_1^nD = hD$ and $g = X^mg_1$. Therefore, $f^nD \cap g^nD = X^{nm}hD$.

Case 3: $f = X^kf_1$ and $g = X^mg_1$, where $m \geqslant k \geqslant 2$, $f_1 \in B[X]$ with $f_1(0) \neq 0$, and $g_1 \in B[X]$ with $g_1(0) \neq 0$. As in the proof of Case 2, we may assume that $f_1, g_1 \in D$. If $k = m$, then there exists an integer $n \geqslant 1$ such that $f_1^nD \cap g_1^nD$ is principal by Case 1. Thus $f^nD \cap g^nD = (X^kf_1)^nD \cap (X^kg_1)^nD = X^{nk}(f_1^nD \cap g_1^nD)$ is principal. If $m > k$, then replacing f and g with f^2 and g^2, we may assume that $m - k \geqslant 2$; so $X^{m-k}g_1 \in D$.

Thus by Case 2, $f_1^n D \cap (X^{m-k}g_1)^n D = hD$ for some integer $n \geqslant 1$ and $h \in D$. Hence $f^n D \cap g^n D = (X^k f_1)^n D \cap (X^m g_1)^n D = X^{nk}(f_1^n D \cap (X^{m-k}g_1)^n D) = X^{nk}hD$.

($\Leftarrow$) Assume that D is an AGCD-domain. Note that if $char\,(A) \neq 0$, then R is a t-linked root extension of D by Lemmas 3.1 and 3.2, and thus R is an AGCD-domain [19, Remark 4.1(b)]. So it suffices to show that $\mathrm{char}(A) \neq 0$.

Let $f = X^2(1+X)$ and $I = (f, 1-X^2)_v \subseteq D$. We first prove that I is t-locally principal, and thus t-invertible by Lemma 2.1(4). Let Q be a maximal t-ideal of D and $S = \{X^n | n = 0, 2, 3, \ldots\}$. If $Q \cap S \neq \emptyset$, then $I \not\subseteq Q$, and so $ID_Q = D_Q$. Next assume that $Q \cap S = \emptyset$. Note that $fD_S \subseteq (f, 1-X^2)_v D_S \subseteq ((f, 1-X^2)_v D_S)_v = ((f, 1-X^2)D_S)_v = (fD_S)_v = fD_S$ by Lemma 2.1(1); so $ID_S = fD_S$. Hence $ID_Q = (ID_S)_{Q_S} = (fD_S)_{Q_S} = fD_Q$.

Recall that an AGCD-domain has a torsion class group [1, Theorem 3.4]. So $(I^n)_t = (((f, 1-X^2)_v)^n)_v = ((f, 1-X^2)^n)_v = hD$ for some $h \in D$ and integer $n \geqslant 1$. Thus $hB[X, X^{-1}] = hD_S = ((f, 1-X^2)^n)_v D_S = ((f, 1-X^2)^n D_S)_v = ((1+X)^n D_S)_v = (1+X)^n D_S = (1+X)^n B[X, X^{-1}]$ (the third equality follows from Lemma 2.1(1) because $(f, 1-X^2)^n$ is t-invertible), and so $h = uX^m(1+X)^n$ for some $u \in U(B)$ and integer m. But since $(1-X^2)^n \in hD$, we have $h(0) \neq 0$, and so $m = 0$. Hence $h = u(1+X)^n = u + unX + \ldots + uX^n$, and thus $nX = 0$ because $h \in D$ and $u \in U(B)$. This means that $char\,(A) \neq 0$. $\square$

Let A be an integrally closed AGCD-domain with $char\,(A) \neq 0$. Then $A[X]$ is an AGCD-domain. So by Theorem 3.3 and [8, Corollary 7], $A[X^2, X^3]$ is a non-integrally closed AGCD-domain with $Cl(A[X^2, X^3]) = Cl(A) \oplus K$, where K, the quotient field of A, is considered as an additive abelian group. It is interesting to note here that $Cl(A[X^2, X^3])$ is torsion.

Corollary 3.4. *Let A be an integral domain, S a saturated multiplicative subset of A, X an indeterminate over A, $R = A + XA_S[X]$, and $D = A + X^2A_S[X]$. Then the following statements are equivalent*:

(1) *D is an AGCD-domain.*
(2) *R is an AGCD-domain and $char\,(A) \neq 0$.*
(3) *A and $A_S[X]$ are AGCD-domains, $char\,(A) \neq 0$, and S is an almost splitting of A.*
(4) *A is an AGCD-domain, $A_S[X] \subseteq A'_S[X]$ is a root extension, and $char\,(A) \neq 0$, where A' is the integral closure of A.*

Proof. (1) $\Leftrightarrow$ (2): This is Theorem 3.3. (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4): See [6, Theorem 3.10]. $\square$

Corollary 3.5. *Let B be an overring of an integral domain A, X an indeterminate over A, $R = A + XB[X]$, and $D = A + X^2B[X]$. Then the following statements are equivalent*:

(1) *D is an AGCD-domain with integral closure R.*
(2) *R is an integrally closed AGCD-domain and $char\,(A) \neq 0$.*
(3) *A is an integrally closed AGCD-domain, $char\,(A) \neq 0$, and $B = A_S$, where S is an almost splitting set of A.*

Proof. (1) $\Leftrightarrow$ (2): This follows directly from Theorem 3.3 because R is integral over D. (2) $\Leftrightarrow$ (3): See [19, Theorem 3.1(b)]. $\square$

In [19], the authors studied integrally closed AGCD-domain of finite t-character of the form $A+XB[X]$ and constructed non-integrally closed AGCD-domains of finite t-character using local algebraic techniques. The following corollary gives many simple examples of non-integrally closed AGCD-domains of finite t-character.

Corollary 3.6. *Let A be an AGCD-domain with $char\,(A) \neq 0$, X an indeterminate over A, S an almost splitting set of A, and $D = A + X^2A_S[X]$. Then D is an AGCD-domain of finite t-character if A is an integrally closed AGCD-domain of finite t-character and S does not contain any infinite sequence of mutually v-coprime nonunit elements.*

Proof. By Dumitrescu et al. [19, Theorem 3.1], $R = A + XA_S[X]$ is an integrally closed AGCD-domain of finite t-character. So D is an AGCD-domain of finite t-character by Lemma 3.1 and Corollary 3.5. $\square$

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